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THE SYMMETRIC MELLIN TRANSFORM IN QUANTUM CALCULUS

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In this paper, we define the q -analogue of Mellin transform symmetric under interchange of q and q^{-1} , and present some of its main properties and explore the possibility of using the integral transform to solve a class of differential q -differences equations.

1. Introduction

The study of q -analysis is an old subject, which dates back to the end of the 19th century. The subject of q -analysis concerns mainly the properties of the so-called q -special functions, which are the extensions of the classical special functions based on a parameter, or the base, q .

It is well known that one of the purposes of integral transforms like Fourier, and Mellin is to solve differential equations using these q -special functions.

In this present paper, we are concerned with the study of the q -analogue of the Mellin transform using the symmetric q -Jackson integral. We also discuss its properties and we give its inversion formula. Furthermore, a q -analogue of the Titchmarsh theorem is proved and we solve respectively the q -diffusion and the q -wave equations using the symmetric q -derivative operator and the symmetric q -Mellin transform.

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This paper is organized as follows. In Section 2, we present some preliminary results and notations useful in the following sections. In Section 3 we introduce the symmetric q -Mellin transform, we discuss its definition domain and its properties. Special attention is devoted to the inversion formula. In Section 4 we study the q -analogue of the convolution product. Finally, in Section 5, a q -analogue of the Titchmarsh theorem and a solutions of the q -diffusion and the q -wave equations are given.

2. Notations and preliminaries

Throughout this paper, we will fix $q > 0$, $q \neq 1$. We recall some usual notions and notations used in q -theory (see [1–3, 5, 7–11]).

Let $a \in \mathbb{C}$, the symmetric q -numbers $\widetilde{[a]}_q$ and symmetric q -factorials $\widetilde{[n]}_q!$ are defined by

$$\widetilde{[a]}_q = \frac{q^a - q^{-a}}{q - q^{-1}}, \quad (1)$$

and

$$\widetilde{[n]}_q! = \widetilde{[1]}_q \widetilde{[2]}_q \cdots \widetilde{[n]}_q. \quad (2)$$

Clearly, these two symmetric q -analogues satisfy

$$\widetilde{[a]}_q = q^{-a+1} [a]_{q^2} \quad \text{and} \quad \widetilde{[n]}_q! = q^{\frac{-n(n-1)}{2}} [n]_{q^2}!, \quad (3)$$

where

$$[a]_q = \frac{1 - q^a}{1 - q}. \quad (4)$$

It is easy to prove that

$$\begin{aligned} \widetilde{[n]}_q &= \widetilde{[n]}_{q^{-1}} = -\widetilde{[-n]}_q, \\ \widetilde{[n+m]}_q &= q^n \widetilde{[m]}_q + q^{-m} \widetilde{[n]}_q = q^m \widetilde{[n]}_q + q^{-n} \widetilde{[m]}_q \\ \widetilde{[0]}_q &= 0, \widetilde{[1]}_q = 1. \end{aligned} \quad (5)$$

The symmetric q -shifted factorial is defined by (see [11]):

$$\left(\widetilde{[a]}_q \right)_m = \begin{cases} \widetilde{[a]}_q \widetilde{[a+1]}_q \cdots \widetilde{[a+m-1]}_q & \text{if } m = 1, 2, \dots, \\ 1 & \text{if } m = 0 \end{cases}, \quad (6)$$

$$= \begin{cases} q^{\frac{-m(m-1)}{2}} \frac{q^{-am}}{(q^{-1}-q)^m} (q^{2a}, q^2)_m & \text{if } m = 1, 2, \dots, \\ 1 & \text{if } m = 0 \end{cases}, \quad (7)$$

where

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{N}^*. \quad (8)$$

The symmetric q -hypergeometric series ${}_n\tilde{\Psi}_{n-1}$ is defined by (see [11]):

$${}_n\tilde{\Psi}_{n-1} \left(\begin{matrix} a_1, \dots, a_n \\ b_1, \dots, b_{n-1} \end{matrix}; q; z \right) = \sum_{m=0}^{\infty} \frac{(\widetilde{[a_1]_q})_m \cdots (\widetilde{[a_n]_q})_m}{(\widetilde{[b_1]_q})_m \cdots (\widetilde{[b_{n-1}]_q})_m} \frac{z^m}{\widetilde{[m]_q}!}. \quad (9)$$

For arbitrary number of numerator and denominator parameters, we introduce the generalized symmetric q -hypergeometric series as:

for $q \in]0, 1[$,

$${}_r\tilde{\Psi}_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q; z \right) = \sum_{m=0}^{\infty} \frac{(\widetilde{[a_1]_q})_m \cdots (\widetilde{[a_r]_q})_m}{(\widetilde{[b_1]_q})_m \cdots (\widetilde{[b_s]_q})_m} \left[\frac{q^{\frac{-m(m-1)}{2}}}{(q^{-1} - q)^m} \right]^{1+s-r} \frac{z^m}{\widetilde{[m]_q}!}, \quad (10)$$

and for $q > 1$,

$${}_r\tilde{\Psi}_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q; z \right) = \sum_{m=0}^{\infty} \frac{(\widetilde{[a_1]_q})_m \cdots (\widetilde{[a_r]_q})_m}{(\widetilde{[b_1]_q})_m \cdots (\widetilde{[b_s]_q})_m} \left[\frac{q^{\frac{m(m-1)}{2}}}{(q - q^{-1})^m} \right]^{1+s-r} \frac{z^m}{\widetilde{[m]_q}!}. \quad (11)$$

We introduce the symmetric q -binomial theorem, expressed with the symmetric q -hypergeometric series by

$${}_1\tilde{\Psi}_0 \left(\begin{matrix} a \\ - \end{matrix}; q; z \right) = \sum_{m=0}^{\infty} \frac{(\widetilde{[a]_q})_m}{\widetilde{[m]_q}!} z^m = \frac{(q^a z; q^2)_{\infty}}{(q^{-a} z; q^2)_{\infty}}. \quad (12)$$

The symmetric q -derivative $\tilde{D}_q f$, of a function f is given by

$$\begin{aligned} \tilde{D}_q f(x) &= \begin{cases} \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}, & \text{if } x \neq 0, \\ f'(0), & \text{when } f'(0) \text{ exists} \end{cases}; \\ &= D_{q^2} f(q^{-1}x), \end{aligned} \quad (13)$$

where

$$D_q f(x) = \begin{cases} \frac{f(qx) - f(x)}{(q-1)x}, & \text{if } x \neq 0, \\ f'(0), & \text{provided } f'(0) \text{ exists} \end{cases}. \quad (14)$$

For $n \in \mathbb{N}$, we note

$$\tilde{D}_q^1 = \tilde{D}_q, \quad \tilde{D}_q^n = \tilde{D}_q (\tilde{D}_q^{n-1}). \quad (15)$$

The symmetric q -derivative has the following property

$$\tilde{D}_q(f(x)g(x)) = f(qx)\tilde{D}_qg(x) + g(q^{-1}x)\tilde{D}_qf(x). \quad (16)$$

The symmetric q -Jackson integrals are defined by (see [2])

$$\begin{aligned} \int_0^a f(x) \tilde{d}_q x &= (q^{-1} - q) a \sum_{n=1,3,\dots}^{+\infty} f(aq^n) q^n, \quad q \in]0, 1[\\ \int_0^\infty f(x) \tilde{d}_q x &= |q^{-1} - q| \sum_{n=\pm 1, \pm 3, \dots} f(q^n) q^n, \end{aligned} \quad (17)$$

and the q -Jackson integrals are given by:

for $q \in]0, 1[$:

$$\int_0^a f(x) d_q x = (1 - q) a \sum_{n=0}^{+\infty} f(aq^n) q^n, \quad (18)$$

$$\int_0^\infty f(x) d_q x = (1 - q) \sum_{n=0}^{+\infty} f(q^n) q^n,$$

$$\int_{-\infty}^{+\infty} f(x) d_q x = (1 - q) \sum_{n=0}^{+\infty} f(q^n) q^n + (1 - q) \sum_{n=0}^{+\infty} f(-q^n) q^n, \quad (19)$$

provided the sums converge absolutely. Using these symmetric q -integrals, we set

$$L_q^1(\tilde{\mathbb{R}}_{q,+}) = \left\{ f : \int_0^\infty |f(x)| \tilde{d}_q x < \infty \right\}, \quad (20)$$

where $\tilde{\mathbb{R}}_{q,+}$ is the set

$$\tilde{\mathbb{R}}_{q,+} = \{q^{2n+1}, \quad n \in \mathbb{Z}\}, \quad (21)$$

and we write for $p > 0$,

$$L_q^1(\mathbb{R}_{q,+}) = \left\{ f : \int_0^{+\infty} |f(x)| d_q x < \infty \right\}, \quad (22)$$

$$L_q^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,q} = \left(\int_{-\infty}^{+\infty} |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\},$$

where

$$\mathbb{R}_{q,+} = \{q^n, \quad n \in \mathbb{Z}\}, \quad \mathbb{R}_q = \{\pm q^n, \quad n \in \mathbb{Z}\}. \quad (23)$$

The symmetric improper q -integral is defined by

$$\int_0^{\infty/A} f(x) \tilde{d}_q x = |q^{-1} - q| \sum_{k \in \mathbb{Z}} f\left(\frac{q^{2k+1}}{A}\right) \frac{q^{2k+1}}{A}. \quad (24)$$

In the case $A = q^{2n}$, we can write

$$\int_0^{\infty/q^{2n}} f(x) \tilde{d}_q x = \int_0^{\infty} f(x) \tilde{d}_q x. \quad (25)$$

The symmetric q -Jackson integral in a generic interval $[a, b]$ is given by

$$\int_a^b f(x) \tilde{d}_q x = \int_0^b f(x) \tilde{d}_q x - \int_0^a f(x) \tilde{d}_q x, \quad (26)$$

we have, in particular

$$\int_{q^{m+1}}^{q^{m-1}} f(x) \tilde{d}_q x = (q^{-1} - q) q^m f(q^m). \quad (27)$$

Theorem 2.1. 1. If F is any anti q -derivative of the function f , namely $\tilde{D}_q F = f$, continuous at $x = 0$, then

$$\int_0^a f(x) \tilde{d}_q x = F(a) - F(0). \quad (28)$$

2. For any function f we have:

$$\tilde{D}_q \left(\int_0^x f(t) \tilde{d}_q t \right) = f(x). \quad (29)$$

3. A symmetric q -analogue of the integration by parts formula is given by

$$\begin{aligned} \int_0^a f(qx) \tilde{D}_q g(x) \tilde{d}_q x &= f(b) g(b) - f(a) g(a) \\ &+ \int_0^a g(q^{-1}x) \tilde{D}_q f(x) \tilde{d}_q x. \end{aligned} \quad (30)$$

The symmetric q -analogues of the exponential function are given by

$$\tilde{E}_q^z = \begin{cases} {}_1\tilde{\Psi}_1 \left(\begin{matrix} 1 \\ 1 \end{matrix}; (q^{-1} - q)z \right) = \sum_{n \geq 0} q^{-\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!} & \text{for } 0 < q < 1 \\ {}_1\tilde{\Psi}_1 \left(\begin{matrix} 1 \\ 1 \end{matrix}; (q - q^{-1})z \right) = \sum_{n \geq 0} q^{\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!} & \text{for } q > 1 \end{cases} \quad (31)$$

$$\tilde{e}_q^z = {}_1\tilde{\Psi}_0(1; -, q; z) = \sum_{n \geq 0} \frac{z^n}{[n]_q!}, \quad (32)$$

and

$$\tilde{\xi}_q^z = \begin{cases} {}_2\tilde{\Psi}_0\left(\begin{smallmatrix} 1, 1 \\ - \end{smallmatrix}; q; \frac{z}{q^{-1}-q}\right) = \sum_{n \geq 0} q^{\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!} & \text{for } 0 < q < 1 \\ {}_2\tilde{\Psi}_0\left(\begin{smallmatrix} 1, 1 \\ - \end{smallmatrix}; q; \frac{z}{q-q^{-1}}\right) = \sum_{n \geq 0} q^{-\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!} & \text{for } q > 1 \end{cases}.$$

The symmetric q -Gamma function $\tilde{\Gamma}_q$ is defined by (see [3])

$$\tilde{\Gamma}_q(x) = q^{-(x-1)(x-2)/2} \Gamma_{q^2}(x), \quad x \neq 0, -1, -2, \dots, \quad (33)$$

where

$$\Gamma_q(x) = \begin{cases} (1-q)^{1-x} \frac{(q; q)_\infty}{(q^x; q)_\infty} & \text{for } q \in]0, 1[\\ q^{\frac{x(x-1)}{2}} (q-1)^{1-x} \frac{(q^{-1}; q^{-1})_\infty}{(q^{-x}; q^{-1})_\infty} & \text{for } q > 1 \end{cases}. \quad (34)$$

It is well known that it satisfies

$$\tilde{\Gamma}_q(x+1) = [\widetilde{x}]_q \tilde{\Gamma}_q(x), \quad \tilde{\Gamma}_q(1) = 1 \quad \text{and} \quad \tilde{\Gamma}_{q^{-1}}(x) = \tilde{\Gamma}_q(x). \quad (35)$$

Theorem 2.2. *For any $x > 0$ we have:*

For $q \in]0, 1[$,

$$\tilde{\Gamma}_q(x) = q^{-x(x-3)/2} K_{q^2}(x) \int_0^{\infty/(q^{-1}-q)} t^{x-1} \tilde{E}_q^{-t} \tilde{d}_q t, \quad (36)$$

where

$$K_{q^2}(x) = \frac{(-q^2, -1; q^2)_\infty}{(-q^{2x}, -q^{2(1-x)}; q^2)_\infty}, \quad (37)$$

and

$$\tilde{\Gamma}_q(x) = q^{-x(x+1)/2} \int_0^{\infty/(q^{-1}-q)} t^{x-1} \tilde{\xi}_q^{-t} \tilde{d}_q t. \quad (38)$$

Moreover, if $\text{Log}(q^{-1} - q) / \text{Log}(q) \in 2\mathbb{Z}$, we obtain

$$\tilde{\Gamma}_q(x) = q^{\frac{-x(x+1)}{2}} \int_0^\infty t^{x-1} \tilde{\xi}_q^{-t} \tilde{d}_q t, \quad q \in]0, 1[. \quad (39)$$

Recently, R. L. Rubin [9] introduced a q -derivative operator ∂_q as follows

$$\partial_q(f)(z) = \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z}. \quad (40)$$

The q^2 -analogue of exponential function is given by

$$e(z; q^2) = \cos(-iz; q^2) + i \sin(-iz; q^2), \quad (41)$$

where cosine and sine are the q -trigonometric functions defined by:

$$\begin{aligned} \cos(x; q^2) &= \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n}}{[2n]_q!}, \\ \sin(x; q^2) &= \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{x^{2n+1}}{[2n+1]_q!}. \end{aligned} \quad (42)$$

In [9], R.L. Rubin defined the q^2 -analogue Fourier transform by

$$\hat{f}(x; q^2) = F_q(f)(x) = K \int_{-\infty}^{+\infty} f(t) e(-itx; q^2) d_q t, \quad x \in \mathbb{R}_q, \quad (43)$$

where

$$K = \frac{(q; q^2)_{\infty}}{2(q^2; q^2)_{\infty}(1-q)^{1/2}}. \quad (44)$$

We remind the following properties:

1. If $f(u), uf(u) \in L_q^1(\mathbb{R}_q)$, then

$$\partial_q(F_q(f))(x) = F_q(-iuf(u))(x). \quad (45)$$

2. If f and $\partial_q f \in L_q^1(\mathbb{R}_q)$, then

$$F_q(\partial_q f)(x) = ix F_q(f)(x). \quad (46)$$

3. For $f \in L_q^2(\mathbb{R}_q)$, we have

$$f(t) = K \int_{-\infty}^{+\infty} F_q(f)(x) e(itx; q^2) d_q x, \quad t \in \mathbb{R}_q. \quad (47)$$

3. The symmetric q -Mellin transform

Definition 3.1. Let f be a function defined on $\mathbb{R}_{q,+}$ we define the symmetric q -Mellin transform of f as

$$\tilde{M}_q(f)(s) = \tilde{M}_q[f(t)](s) = \int_0^{\infty} t^{s-1} f(t) \tilde{d}_q t \quad q \in]0, 1[. \quad (48)$$

Theorem 3.2. *Let f be a function defined on $\tilde{\mathbb{R}}_{q,+}$ and let $u, v \in \mathbb{R}$ with $u > v$. We suppose*

$$f(x) = O_{o+}(x^u) \quad \text{and} \quad f(x) = O_{+\infty}(x^v). \quad (49)$$

Then $\tilde{M}_q(f)(s)$ exists in the strip $\langle -u, -v \rangle$.

Theorem 3.3. *If f is a function defined on $\tilde{\mathbb{R}}_{q,+}$, then $\tilde{M}_q(f)(s)$ is analytic on the strip $\langle \alpha_{q,f}, \beta_{q,f} \rangle$ and we have*

$$\forall s \in \langle \alpha_{q,f}, \beta_{q,f} \rangle, \quad \frac{d}{ds} \tilde{M}_q(f)(s) = \tilde{M}_q[\text{Log}(t)f(t)](s). \quad (50)$$

3.1. Properties

In the following subsection, we give some interesting properties of the symmetric q -Mellin transform.

1. For $a = q^{2n}$, $n \in \mathbb{Z}$ and $s \in \langle \alpha_{q,f}, \beta_{q,f} \rangle$, we have

$$\tilde{M}_q[f(at)](s) = a^{-s} \tilde{M}_q(f)(s). \quad (51)$$

2. For $s \in \langle -\beta_{q,f}, -\alpha_{q,f} \rangle$, we have

$$\tilde{M}_q \left[f \left(\frac{1}{t} \right) \right] (s) = \tilde{M}_q(f)(-s). \quad (52)$$

3. For $s \in \langle 1 - \beta_{q,f}, 1 - \alpha_{q,f} \rangle$, we have

$$\tilde{M}_q \left[\frac{1}{t} f \left(\frac{1}{t} \right) \right] (s) = \tilde{M}_q(f)(1-s). \quad (53)$$

4. For $s \in \langle \alpha_{q,f}, \beta_{q,f} \rangle$, we have

$$\tilde{M}_q[t\tilde{D}_{q^2}f(t)](s) = \widetilde{[-s]}_{q^2} \tilde{M}_q(f)(s). \quad (54)$$

5. For $s \in \langle \alpha_{q,f} + 1, \beta_{q,f} + 1 \rangle$, we have

$$\tilde{M}_q[\tilde{D}_{q^2}f(t)](s) = \widetilde{[1-s]}_{q^2} \tilde{M}_q(f)(s-1). \quad (55)$$

By induction, we obtain that, for $n \in \mathbb{N}^*$ and $s \in \langle \alpha_{q,f} + n, \beta_{q,f} + n \rangle$,

$$\tilde{M}_q[\tilde{D}_{q^2}^{(n)}f(t)](s) = \widetilde{[1-s]}_{q^2} \dots \widetilde{[n-s]}_{q^2} \tilde{M}_q(f)(s-n).$$

6. Given ρ a positive odd integer and $s \in \langle \rho \alpha_{q^\rho, f}, \rho \beta_{q^\rho, f} \rangle$, we have

$$\tilde{M}_q[f(t^\rho)](s) = \left[\frac{1}{\rho} \right]_{q^\rho} \tilde{M}_{q^\rho}(f)\left(\frac{s}{\rho}\right).$$

7. Let $(\mu_k)_k$ be a sequence of $\mathbb{R}_{q,+} \setminus \tilde{\mathbb{R}}_{q,+}$, let $(\lambda_k)_k$ be a sequence of \mathbb{C} , and let f be a suitable function, then we have

$$\tilde{M}_q \left[\sum_{k \geq 0} \lambda_k f(\mu_k t) \right] (s) = \left(\sum_{k \geq 0} \frac{\lambda_k}{\mu_k^s} \right) \tilde{M}_q(f)(s),$$

provided the sums converge.

3.2. The symmetric q -Mellin inversion formula

Theorem 3.4. Let f be a function defined over $\tilde{\mathbb{R}}_{q,+}$, and let $c \in]\alpha_{q,f}, \beta_{q,f}[$, then

$$\forall x \in \tilde{\mathbb{R}}_{q,+}, \quad \frac{1}{2i\pi} \frac{\log(q)}{(q^{-1} - q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \tilde{M}_q(f)(s) x^{-s} ds = f(x). \quad (56)$$

Proof. Let $c \in]\alpha_{q,f}, \beta_{q,f}[$ and $x = q^{2k+1} \in \tilde{\mathbb{R}}_{q,+}$, we have:

$$\begin{aligned} & \frac{1}{2i\pi} \frac{\log(q)}{(q^{-1} - q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \tilde{M}_q(f)(s) x^{-s} ds \\ &= \frac{\log(q)}{2i\pi} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \left(\sum_{n \in \mathbb{Z}} f(q^{2n+1}) (q^{2n+1})^s \right) (q^{2k+1})^{-s} ds \\ &= \frac{\log(q)}{2i\pi} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \left(\sum_{n \in \mathbb{Z}} f(q^{2n+1}) (q^{2n-2k})^s \right) ds, \end{aligned}$$

since the series $\sum_{n \in \mathbb{Z}} f(q^{2n+1}) (q^{2n-2k})^s$ converge uniformly with respect to s , one gets:

$$\begin{aligned} & \frac{1}{2i\pi} \frac{\log(q)}{(q^{-1} - q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \tilde{M}_q(f)(s) x^{-s} ds \\ &= \frac{i \log(q)}{2i\pi} \sum_{n \in \mathbb{Z}} q^{2c(n-k)} f(q^{2n+1}) \int_{-\pi/\log(q)}^{\pi/\log(q)} q^{2i(n-k)t} dt \\ &= \sum_{n \in \mathbb{Z}} q^{2c(n-k)} f(q^{2n+1}) \delta_{n,k} \\ &= f(q^{2k+1}). \end{aligned}$$

□

3.3. Symmetric q -Mellin's convolution product

Definition 3.5. The symmetric q -Mellin convolution product of the functions f and g is the function $f \star_{\tilde{M}_q} g$ defined by

$$f \star_{\tilde{M}_q} g(x) = \int_0^\infty f(t) g\left(q \frac{x}{t}\right) \frac{\tilde{d}_q t}{t}, \quad x \in \tilde{\mathbb{R}}_{q,+}, \quad (57)$$

provided the symmetric q -integral exists.

Theorem 3.6. *If the symmetric q -Mellin convolution product of f and g exists, then*

$$f \star_{\tilde{M}_q} g = g \star_{\tilde{M}_q} f, \quad (58)$$

$$\tilde{M}_q \left[f \star_{\tilde{M}_q} g \right] (s) = q^{-s} \tilde{M}_q(f)(s) \tilde{M}_q(g)(s). \quad (59)$$

Proof.

$$\begin{aligned} \tilde{M}_q \left[f \star_{\tilde{M}_q} g \right] (s) &= \int_0^\infty x^{s-1} f \star_{\tilde{M}_q} g(x) \tilde{d}_q x \\ &= (q^{-1} - q) \int_0^\infty x^{s-1} \left(\sum_{n \in \mathbb{Z}} f(q^{2n+1}) g(xq^{-2n}) \right) \tilde{d}_q x \\ &= (q^{-1} - q) \sum_{n \in \mathbb{Z}} f(q^{2n+1}) \int_0^\infty x^{s-1} g(xq^{-2n}) \tilde{d}_q x \\ &= (q^{-1} - q) \sum_{n \in \mathbb{Z}} f(q^{2n+1}) (q^{-2n})^{-s} \tilde{M}_q(g)(s) \\ &= q^{-s} \tilde{M}_q(g)(s) \tilde{M}_q(f)(s). \quad \square \end{aligned}$$

Theorem 3.7. *For the suitable functions f and g , we have the following relations:*

$$\frac{1}{2i\pi} \frac{\log(q)}{(q^{-1} - q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \tilde{M}_q(f)(s) \tilde{M}_q(g)(1-s) ds = \int_0^\infty g(x) f(x) \tilde{d}_q x, \quad (60)$$

and

$$\frac{1}{2i\pi} \frac{\log(q)}{(q^{-1} - q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} q^{-s} \tilde{M}_q(f)(s) \tilde{M}_q(g)(s) ds = \int_0^\infty f(t) g\left(\frac{q}{t}\right) \frac{\tilde{d}_q t}{t}. \quad (61)$$

Proof. In order to prove the first relation, let $c \in \mathbb{R}$ such that $c \in]\alpha_{q,f}, \beta_{q,f}[\cap]1 - \beta_{q,g}, 1 - \alpha_{q,g}[$. We put $I(c) = [c - i\pi/\log(q), c + i\pi/\log(q)]$. From the symmetric q -Mellin inversion formula, and the relation

$$\sup_{s \in I(c)} \left| (q^{2n+1})^{1-s} g(q^{2n+1}) \tilde{M}_q(f)(s) \right| = q^{(1-c)(2n+1)} |g(q^{2n+1})| \sup_{s \in I(c)} |\tilde{M}_q(f)(s)|,$$

we obtain

$$\begin{aligned}
 & \frac{1}{2i\pi} \frac{\log(q)}{(q^{-1}-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \tilde{M}_q(f)(s) \tilde{M}_q(g)(1-s) ds \\
 &= \frac{1}{2i\pi} \log(q) \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \sum_{n \in \mathbb{Z}} (q^{2n+1})^{1-s} g(q^{2n+1}) \tilde{M}_q(f)(s) ds \\
 &= \frac{\log(q)}{2i\pi} \sum_{n \in \mathbb{Z}} q^{2n+1} g(q^{2n+1}) \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} (q^{2n+1})^{-s} \tilde{M}_q(f)(s) ds \\
 &= \frac{1}{2i\pi} \frac{\log(q)}{(q^{-1}-q)} \int_0^\infty g(x) \left(\int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} x^{-s} \tilde{M}_q(f)(s) ds \right) \tilde{d}_q x \\
 &= \int_0^\infty g(x) f(x) \tilde{d}_q x. \quad \square
 \end{aligned}$$

4. Applications

4.1. Symmetric q -integral equations

Theorem 4.1. *Let K and g be functions defined on $\tilde{\mathbb{R}}_{q,+}$. We suppose that $\langle \alpha_{q,L}, \beta_{q,L} \rangle \cap \langle 1 - \beta_{q,g}, 1 - \alpha_{q,g} \rangle$ is not empty, for a suitable real c , we put*

$$\forall x \in \tilde{\mathbb{R}}_{q,+}, L(x) = \frac{q^{-1}}{2i\pi} \frac{\log(q)}{(q^{-1}-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \frac{x^{-s}}{\tilde{M}_q(K)(1-s)} ds. \quad (62)$$

Then the q -symmetric integral equation

$$\int_0^\infty f(x) K(q^{-1}xt) \tilde{d}_q x = g(t), \quad t \in \tilde{\mathbb{R}}_{q,+}, \quad (63)$$

has as solution

$$f(x) = \int_0^\infty g(t) L(q^{-1}xt) \tilde{d}_q t, \quad x \in \tilde{\mathbb{R}}_{q,+}. \quad (64)$$

In addition, if

$$\tilde{M}_q(K)(s) \tilde{M}_q(K)(1-s) = q^{-1}, \quad (65)$$

then equation (63) has the solution:

$$f(x) = \int_0^\infty g(t) K(q^{-1}xt) \tilde{d}_q t, \quad x \in \tilde{\mathbb{R}}_{q,+}. \quad (66)$$

Proof. By taking the symmetric q -Mellin transform in the equation (63), we obtain

$$\tilde{M}_q(f)(s) \tilde{M}_q(K)(1-s) = q^{s-1} \tilde{M}_q(g)(1-s).$$

From the relation (62), we deduce that

$$\tilde{M}_q(f)(s) = q^s \tilde{M}_q(L)(s) \tilde{M}_q(g)(1-s).$$

Then, for $c' \in \langle \alpha_{q,L}, \beta_{q,L} \rangle \cap \langle 1 - \beta_{q,g}, 1 - \alpha_{q,g} \rangle$, we get:

$$\forall x \in \tilde{\mathbb{R}}_{q,+},$$

$$\begin{aligned} f(x) &= \frac{\log(q)}{2i\pi(q^{-1}-q)} \int_{c'-i\pi/\log(q)}^{c'+i\pi/\log(q)} \tilde{M}_q(L)(s) \tilde{M}_q(g)(1-s) (q^{-1}x)^{-s} ds \\ &= \frac{\log(q)}{2i\pi(q^{-1}-q)} \int_{c'-i\pi/\log(q)}^{c'+i\pi/\log(q)} \tilde{M}_q[L(q^{-1}xt)](s) \tilde{M}_q(g)(1-s) ds. \end{aligned}$$

Finally, by the relation (60), we have:

$$\forall x \in \tilde{\mathbb{R}}_{q,+}, \quad f(x) = \int_0^\infty g(t) L(q^{-1}xt) \tilde{d}_q t. \quad \square$$

4.2. Symmetric q -analogue of the Titchmarsh theorem

Theorem 4.2. *Let K be a function defined on $\tilde{\mathbb{R}}_{q,+}$. Suppose that $\langle \alpha_{q,K}, \beta_{q,K} \rangle$ is not empty. If the integral equation*

$$f(x) = \int_0^\infty K(q^{-1}xu) \tilde{d}_q u \int_0^\infty K(q^{-1}yu) f(y) \tilde{d}_q y, \quad (67)$$

has a suitable solution f then, for every $s \in \mathbb{C}$ such that s and $1-s$ are in $\langle \alpha_{q,K}, \beta_{q,K} \rangle$, we have

$$\tilde{M}_q(K)(s) \tilde{M}_q(K)(1-s) = q^{-1}.$$

Proof. The integral equation (67) is written as a pair of reciprocal formulas as follows

1. $g(x) = \int_0^\infty K(q^{-1}yx) f(y) \tilde{d}_q y,$
2. $f(x) = \int_0^\infty K(q^{-1}yx) g(y) \tilde{d}_q y.$

By taking the symmetric q -Mellin transform in (1) and (2) at s , we get

$$\tilde{M}_q(g)(s) = q^s \tilde{M}_q(f)(1-s) \tilde{M}_q(K)(s),$$

and

$$\tilde{M}_q(f)(s) = q^s \tilde{M}_q(g)(1-s) \tilde{M}_q(K)(s),$$

changing s into $1-s$ in one of these equations and multiplying, we deduce that

$$\tilde{M}_q(K)(s) \tilde{M}_q(K)(1-s) = q^{-1}. \quad \square$$

4.3. Symmetric q -diffusion equation

We assume that $\frac{\text{Log}(q^{-1}-q)}{\text{Log}(q)} \in 2\mathbb{Z}$. We consider the following symmetric q -diffusion equation:

$$\tilde{D}_{q^2,t} u(x,t) = (\partial_{q,x})^2 u(x, q^2 t), \quad x \in \mathbb{R}_q \text{ and } t \in \tilde{\mathbb{R}}_{q,+}, \quad (68)$$

subject to the initial condition

$$u(x, 0) = f(x), \quad f \in L_q^2(\mathbb{R}_q). \quad (69)$$

By taking a the q^2 -Fourier transform in x and the symmetric q -Mellin transform in t , we obtain

$$\widetilde{[s-1]}_q U(\lambda, s-1) = \lambda^2 q^{-2s} U(\xi, s). \quad (70)$$

The general solution is [6]

$$U(\lambda, s) = C(\lambda) \lambda^{-2s} q^{s(s+1)} \tilde{\Gamma}_{q^2}(s), \quad (71)$$

where $C(\lambda)$ is a function of λ only. For the relation (39), the inversion symmetric q -Mellin transform of $\lambda^{-2s} q^{s(s+1)} \tilde{\Gamma}_{q^2}(s)$ is

$$\frac{\log(q)}{2i\pi(q^{-1}-q)} \int_{c-i\pi/\log(q)}^{c+i\pi/\log(q)} \lambda^{-2s} q^{s(s+1)} \tilde{\Gamma}_{q^2}(s) t^{-s} ds = \tilde{\xi}_{q^2}^{-\lambda^2 t}. \quad (72)$$

Then

$$u(x, t) = K \int_{-\infty}^{+\infty} C(\lambda) \tilde{\xi}_{q^2}^{-\lambda^2 t} e(i\lambda x, q^2) d_q \lambda. \quad (73)$$

For $t = 0$, we get

$$u(x, 0) = K \int_{-\infty}^{+\infty} C(\lambda) e(i\lambda x, q^2) d_q \lambda = f(x), \quad (74)$$

so

$$C(\lambda) = K \int_{-\infty}^{+\infty} f(x) e(-i\lambda x, q^2) d_q x = \hat{f}(\lambda, q^2). \quad (75)$$

Therefore, a solution of (68) is

$$u(x, t) = K \int_{-\infty}^{+\infty} \hat{f}(\lambda, q^2) \tilde{\xi}_{q^2}^{-\lambda^2 t} e(i\lambda x, q^2) d_q \lambda. \quad (76)$$

4.4. Symmetric q -wave equation

We assume that $\frac{\text{Log}(q^{-1}-q)}{\text{Log}(q)} \in 2\mathbb{Z}$. Let's consider the following symmetric q -wave equation:

$$(\tilde{D}_{q^2,t})^2 u(x, q^{-2}t) = (\partial_{q,x})^2 u(x, q^2t), \quad x \in \mathbb{R}_q \text{ and } t \in \tilde{\mathbb{R}}_{q,+}, \quad (77)$$

with the initial conditions

$$u(x, 0) = f(x), \quad \tilde{D}_{q^2,t} u(x, 0) = g(x), \quad f, g \in L_q^2(\mathbb{R}_q). \quad (78)$$

By applying the q^2 -Fourier and the symmetric q -Mellin transform, we obtain

$$\widetilde{[s-1]_{q^2}} \widetilde{[s-2]_{q^2}} U(\lambda, s-2) = -\lambda^2 q^{-2(2s-1)} U(\lambda, s). \quad (79)$$

A solution of the equation (79) is given by:

$$U(\lambda, s) = [C(\lambda)(-i\lambda)^{-s} + C'(\lambda)(i\lambda)^{-s}] q^{s(s+1)} \tilde{\Gamma}_{q^2}(s), \quad (80)$$

where $C(\lambda)$ and $C'(\lambda)$ are functions of λ only.

From the symmetric q -Mellin inversion formula, we get

$$\hat{u}(\lambda, t) = \left[C(\lambda) \tilde{\xi}_{q^2}^{i\lambda t} + C'(\lambda) \tilde{\xi}_{q^2}^{-i\lambda t} \right], \quad (81)$$

where $\hat{u}(\lambda, t)$ is the q^2 -Fourier transform of $u(x, t)$ with respect to x .

Now we rewrite (81) in terms of the symmetric q -Sine and the q -Cosine function which are defined by

$$\widetilde{\text{Sin}}_q(x) = \frac{\tilde{\xi}_q^{ix} - \tilde{\xi}_q^{-ix}}{2i}, \quad \widetilde{\text{Cos}}_q(x) = \frac{\tilde{\xi}_q^{ix} + \tilde{\xi}_q^{-ix}}{2}. \quad (82)$$

The result is

$$\hat{u}(\lambda, t) = D(\lambda) \widetilde{\text{Cos}}_{q^2}(\lambda t) + D'(\lambda) \widetilde{\text{Sin}}_{q^2}(\lambda t), \quad (83)$$

where $D(\lambda)$ and $D'(\lambda)$ are functions of λ .

Now, the inverse q^2 -Fourier transform of (83) gives:

$$u(x, t) = K \int_{-\infty}^{+\infty} \left(D(\lambda) \widetilde{\text{Cos}}_{q^2}(\lambda t) + D'(\lambda) \widetilde{\text{Sin}}_{q^2}(\lambda t) \right) e(i\lambda x, q^2) d_q \lambda. \quad (84)$$

By taking $t = 0$ in (84), we get $D(\lambda) = \hat{f}(\lambda, q^2)$.

On the other hand, by using the relations

$$\tilde{D}_{q^2,t} \widetilde{\text{Sin}}_{q^2}(\lambda t) = -\lambda \widetilde{\text{Sin}}_{q^2}(q^2 \lambda t),$$

and

$$\tilde{D}_{q^2,t} \widetilde{\text{Cos}}_{q^2}(\lambda t) = \lambda \widetilde{\text{Cos}}_{q^2}(q^2 \lambda t),$$

we get

$$\hat{g}(\lambda, q^2) = D'(\lambda) \lambda. \quad (85)$$

Therefore the final solution of (77) is

$$u(x, t) = K \int_{-\infty}^{+\infty} \left(\hat{f}(\lambda, q^2) \widetilde{\text{Cos}}_{q^2}(\lambda t) + \frac{\hat{g}(\lambda, q^2)}{\lambda} \widetilde{\text{Sin}}_{q^2}(\lambda t) \right) e(i\lambda x, q^2) d_q \lambda. \quad (86)$$

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